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Heavy quark induced effective action for gauge fields in the $SU(N_c) \otimes U(1)$ model and the low-energy structure of heavy quark current correlators

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Abstract. We calculate the low-energy limit of heavy quark current correlators within an expansion in the inverse heavy quark mass. The induced low-energy currents built from the gluon fields corresponding to the initial heavy quark currents are obtained from an effective action for gauge fields in the one-loop approximation at the leading order of the $1/m$ expansion. Explicit formulae for the low-energy spectra of electromagnetic and tensor heavy quark current correlators are given. Consequences of the appearance of a nonvanishing spectral density below the two-particle threshold for high precision phenomenology of heavy quarks are discussed quantitatively.

1 Introduction

Quantum corrections can qualitatively change the analytic structure of the Green function or the symmetry properties of a classical field-theoretical system. Qualitatively new features compared to the tree-level picture may emerge already at finite orders of perturbation theory for Green functions while some effects can only appear as a result of summing up an infinite number of perturbative terms. An effect of non-conservation of the Abelian axial current reveals itself at leading order as a result of the calculation of an one-loop triangle diagram [1, 2] while spontaneous symmetry breaking or bound state formation cannot be observed at any finite order of perturbation theory and requires an infinite summation of relevant subsets of diagrams. For problems related to an investigation of the symmetry properties of quantum systems, such an infinite summation can be readily done by introducing an effective action for the system and calculating it as a loop-expansion series. Such an approach reorders the perturbation series with respect to the lines of diagrams related to external fields and allows one to take into account the entire dependence on external fields exactly within a given order of the loop expansion. An effective action can be considered as a generating functional for the vertex (one-particle irreducible, or proper) Green functions (see e.g. [3]). The treatment of external fields beyond the finite order of expansion was done in [4]. The efficient method of calculating the effective action is based on the Legendre transform of the generating functional for connected Green functions and heavily uses the functional techniques [5]. One can also use a practical calculation technique by substituting the shifted fields in the original Lagrangian of the system [6, 7]. The part of the effective action which is constructed from the constant field is usually called an effective potential and is used to analyze the fundamental symmetry properties of the theory beyond the plain perturbation theory where the effect of the external fields is resummed to all orders. The expansion leading to the effective potential is, in fact, an expansion in Planck's constant $\hbar$, i.e. the correction accounts for the deflection from the classical limit. Therefore, new effects which are absent in the classical approximation may appear within such an approach. An example for this kind of new quantum effects is the light-by-light scattering which emerges as a quantum correction to the photon dynamics due to the interaction with virtual electrons. In the low-energy limit it can be seen as a nonlinearity of the equations for the strong electromagnetic fields in the vacuum. The behavior of the electromagnetic fields with such a correction can be described by the Euler–Heisenberg Lagrangian [8]. The generalization to non-Abelian fields is discussed in [9]. The effective potential is also a powerful tool for investigating the effects of spontaneous symmetry breaking by quantum corrections [10] and for analyzing the properties of particle systems at finite temperature and density [11].

A special advantage of using the external field technique in gauge theories is an explicit gauge invariance of the effective action that allows one to drastically simplify the computation and to reduce the number of necessary diagrams [12]. It is generally believed that in non-Abelian gauge theories the nonperturbative fluctuations (instantons) [13] create a complex vacuum structure that eventually explains (or is responsible for) a low-energy spectrum of observed particles [14]. The technique of calculating in external fields was heavily used for the calculation of quantum corrections to the effective action of gauge fields in the classical instanton background [15]. In practical applications to QCD the complex vacuum structure could explain the phase transition from the quark–gluon representation of Green functions at high energies to the hadron picture at low energies. While the problem of a full description of this transition remains to be solved, Wilson's operator product expansion, which is one of the key tools for calculating the correlation functions at short distances, is used for describing hadronic properties at low energies in a semiphenomenological way using sum rule techniques [16]. The external field technique provides a convenient way for practical calculations within the sum rule method based on a semiphenomenological account for the vacuum condensates of local operators [17,18].

In the present paper we calculate the low-energy limit of heavy quark current correlators within the $1/m$ expansion, where m is a heavy quark mass. The induced low-energy currents corresponding to the heavy quark initial operators are obtained from an effective action for gauge fields of the $SU(N_c)\otimes U(1)$ model in the one-loop approximation at leading order of both the coupling constant and the $1/m$ expansion. Our results for the effective action for vector and tensor currents are presented in Sect. 2. Using the external field technique as a convenient framework for practical calculations of the induced currents in this model, in Sect. 3 we present these induced currents. In Sect. 4 we show phenomenological applications, and in Sect. 5 we deal with the consequences encountered for calculating arbitrary moments of the spectral densities of various correlators.

2 The effective action

While the technique is standard, we are aiming at concrete results for further phenomenological applications to sum rules for the vacuum polarization functions of heavy quarks. Therefore, we briefly outline the calculation and the related issues. Further details can be found in [3,19]. The Lagrangian of a heavy fermion field ψ interacting with a gauge field $\mathcal{B}$ of the gauge group $SU(N_c)\otimes U(1)$ reads

$$L=\bar\psi\left(\mathrm{i}\gamma^\mu\partial_\mu+\gamma^\mu\mathcal{B}_\mu-m\right)\psi, \tag{1}$$

where $\mathcal{B}_\mu=eA_\mu+g_{\mathrm{s}}B_\mu$. Here A_μ is a gauge field of the $U(1)$ subgroup (photon) with the coupling constant e and B_μ is a gauge field of the $SU(N_c)$ subgroup (gluon) with the coupling constant g_{s}. The matrix notation for the non-Abelian gauge field potentials is used, $B_\mu=t_aB^a_\mu$, where t_a are generators of the gauge group $SU(N_c)$. A generating functional $W[J]$ of connected Green functions is given by a functional integral with the sources J,

$$\begin{aligned}Z[J]&=\exp(\mathrm{i}W[J])\\&=\int\left[\mathrm{d}\bar\psi\mathrm{d}\psi\right]\exp\left(\mathrm{i}\int L^J(x)\mathrm{d}^4x\right)\\&=\int\left[\mathrm{d}\bar\psi\mathrm{d}\psi\right]\exp\left(\mathrm{i}\int\left(\bar\psi(\mathrm{i}\gamma^\mu\partial_\mu+\gamma^\mu\mathcal{B}_\mu-m)\psi\right.\right.\\&\qquad\left.\left.+J^\mu\mathcal{B}_\mu\right)\mathrm{d}^4x\right),\end{aligned} \tag{2}$$

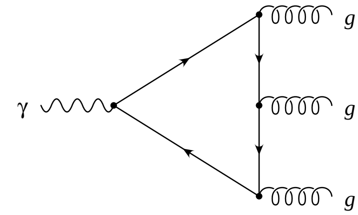

Fig. 1. Heavy quark loop correction to the electromagnetic current

where the product $J^\mu\mathcal{B}_\mu$ implies a trace with respect to the representation of $SU(N_c)\otimes U(1)$. A proper gauge fixing is implied as well. The effective action $\mathit{\Gamma}[\bar{\mathcal{B}}]$ for the gauge field is then given by the Legendre transform

$$\mathit{\Gamma}[\bar{\mathcal{B}}]=W[J]-J\bar{\mathcal{B}},\quad \bar{\mathcal{B}}=\frac{\delta W[J]}{\delta J}. \tag{3}$$

It was shown that this procedure is equivalent to the more direct calculation in external fields (see e.g. [19]). It is also a generalization of results obtained for constant external fields [2]. Up to leading order in $\hbar$ the effective action constructed with a Legendre transform can also be found through

$$\begin{aligned}&\exp(\mathrm{i}\mathit{\Gamma}[\mathcal{B}])\\&\quad=\int\left[\mathrm{d}\bar\psi\mathrm{d}\psi\right]\exp\left(\mathrm{i}\int\bar\psi(\mathrm{i}\gamma^\mu\partial_\mu+\gamma^\mu\mathcal{B}_\mu-m)\psi\mathrm{d}^4x\right)\\&\quad=\det\left(\mathrm{i}\gamma^\mu\partial_\mu+\gamma^\mu\mathcal{B}_\mu-m\right),\end{aligned} \tag{4}$$

where $\mathcal{B}$ is now a classical gauge field (integration over d^4x is implied for expressions of the action). By using the identity $\det M=\exp(\mathrm{tr}(\ln M))$ for an operator M one continues with

$$\mathrm{i}\mathit{\Gamma}[\mathcal{B}]=\mathrm{tr}[\ln\left(\mathrm{i}\gamma^\mu\partial_\mu+\gamma^\mu\mathcal{B}_\mu-m\right)]. \tag{5}$$

2.1 Results for the effective action

A straightforward calculation of the functional determinant gives a correction to the effective low-energy action of the gauge fields within $SU(N_c)\otimes U(1)$. We calculate the leading nontrivial contribution in the $1/m$ expansion. The functional determinant in the loop expansion can be represented by Feynman diagrams. A diagram which gives a correction to the effective action due to a heavy quark loop is shown in Fig. 1. Two-gluon transitions are forbidden according to a generalization of Farry's theorem to non-Abelian theories [20]. We are interested in the behavior of the amplitude associated with the diagram in Fig. 1 at low energies and, therefore, take the limit of a very heavy quark. Formally the limit $m\to\infty$ is taken which in physical terms means that m is much larger than all momenta of the external legs of the diagram in Fig. 1, namely the three gluons and the photon. A straightforward calculation of the diagram presented in Fig. 1 gives the one-loop result for the correction to the effective action induced by a heavy quark loop at leading order of the $1/m$ expansion. This correction reads

$$\Delta\Gamma_{\rm QCD} = \frac{eg_{\rm s}^3 d_{abc}}{180m^4(4\pi)^2} \times \left[14{\rm tr}(FG^aG^bG^c) - 5{\rm tr}(FG^a){\rm tr}(G^bG^c)\right], \quad (6)$$

where $F_{\mu\nu}$ is a field strength tensor for the $U(1)$ subgroup, $G_{\mu\nu} = t_a G^a_{\mu\nu}$ is the field strength tensor for the $SU(N_c)$ subgroup, and d_{abc} are the totally symmetric $SU(N_c)$ structure constants defined by the relation $d_{abc} = 2{\rm tr}(\{t_a, t_b\}t_c)$. The trace in (6) is understood as a trace with respect to the Lorentz indices of the fields, i.e. one considers the field strength tensors of gauge fields as matrices for which ${\rm tr}(FG^a) = F^{\mu\nu}G^a_{\nu\mu}$. This makes the formulae shorter and more transparent. Before we proceed to the calculation of the induced current in the next section, we compare the result with the corresponding expression in QED.

2.2 Comparison with QED

The effective action within QED corresponding to Fig. 1 with gluons substituted by photons is known as the Euler–Heisenberg Lagrangian [19],

$$\Delta\Gamma_{\rm QED} = \frac{2\alpha^2}{45m^4}\left[(\boldsymbol{E}^2 - \boldsymbol{H}^2)^2 + 7(\boldsymbol{E}\cdot\boldsymbol{H})^2\right], \qquad \alpha = \frac{e^2}{4\pi}. \quad (7)$$

This expression can be obtained by a direct calculation in the same way as the result in (6). One can also extract it from (6) by modifying the gauge group factors and taking into account the symmetry of the action with respect to the external gauge fields. We remind the reader that (6) is only the term linear in the photon field, while the higher order contributions are not explicitly written down because they are redundant for our primary purpose of determining the low-energy structure of the heavy quark correlators. In the following we give some relations between the fourth-order monomials of the photon field to convert the basis of (6) into the traditional QED basis used in (7). With the definitions for the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{H}$,

$$\begin{aligned} F^{0j} &= -F_{0j} = -E_j, \\ F^{i0} &= -F_{i0} = E_i, \\ F^{ij} &= F_{ij} = -\epsilon_{ijk}H_k, \end{aligned} \quad (8)$$

one finds

$$\begin{aligned} {\rm tr}(F^2) &= 2(\boldsymbol{E}^2 - \boldsymbol{H}^2), \\ {\rm tr}(F^4) &= 2(\boldsymbol{E}^2 - \boldsymbol{H}^2)^2 + 4(\boldsymbol{E}\cdot\boldsymbol{H})^2, \\ {\rm tr}(F\tilde{F}) &= 4(\boldsymbol{E}\cdot\boldsymbol{H}), \\ {\rm tr}(F\tilde{F}F\tilde{F}) &= -\frac{1}{2}({\rm tr}(F^2))^2 + {\rm tr}(F^4) \\ &= 4(\boldsymbol{E}\cdot\boldsymbol{H})^2, \end{aligned} \quad (9)$$

where

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}. \quad (10)$$

Note for further use and convenience that in order to rewrite the expressions from one form to another one can use one more relation between the fourth-order monomials of the electromagnetic field strength tensor $F_{\mu\nu}$,

$$(F\tilde{F})^2 = -2(F^2)^2 + 4F^4, \quad (11)$$

with

$$\begin{aligned} F\tilde{F} &= F^{\mu\nu}\tilde{F}_{\mu\nu}, \\ F^2 &= F^{\mu\nu}F_{\mu\nu}, \\ F^4 &= F^{\mu\nu}F_{\nu\alpha}F^{\alpha\beta}F_{\beta\mu}. \end{aligned} \quad (12)$$

Therefore the correspondence between (6) and its QED counterpart in (7) is established.

3 The induced current

An expression for the induced electromagnetic current J^μ as being an effective electromagnetic current for the low-energy effective theory describing the interaction of photons and gluons is given by the derivative of the effective action with respect to the external Abelian gauge field,

$$eJ^\mu = -\frac{\delta\Gamma[\mathcal{B}]}{\delta A_\mu} = -{\rm tr}\left[{\rm i}e\gamma_\mu\frac{1}{{\rm i}\gamma^{\mu'}\partial_{\mu'} + \gamma^{\mu'}\mathcal{B}_{\mu'} - m}\right]. \quad (13)$$

The derivative with respect to A_μ can be replaced by a derivative with respect to $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$,

$$eJ^\mu(x) = -\frac{\delta F_{\mu'\nu'}}{\delta A_\mu}\,\frac{\delta\Gamma[\mathcal{B}]}{\delta F_{\mu'\nu'}} = -2\partial_\nu\frac{\delta\Gamma[\mathcal{B}]}{\delta F_{\nu\mu}}. \quad (14)$$

Expressed differently, one can say that after having integrated out the heavy quark field in the functional integral one obtains the relation

$$\langle j^\mu\rangle_\psi = \langle\bar{\psi}\gamma^\mu\psi\rangle_\psi \equiv J^\mu \quad (15)$$

for the electromagnetic current of the heavy quark.

3.1 Results for the induced vector current

With the explicit expression for the effective action given in (6) we obtain [21]

$$J^\mu = \partial_\nu\mathcal{O}^{\mu\nu}, \quad (16)$$

$$\mathcal{O}^{\mu\nu} = \frac{-g_{\rm s}^3 d_{abc}}{90m^4(4\pi)^2}\left[14(G^aG^bG^c)^{\mu\nu} - 5(G^a)^{\mu\nu}{\rm tr}(G^bG^c)\right].$$

Note that the current conservation $\partial_\mu J^\mu = 0$ that inherits the original relation $\partial_\mu j^\mu = 0$ and is necessary for a gauge invariant interaction with photons is automatically guaranteed because the operator $\mathcal{O}^{\mu\nu}$ is antisymmetric, $\mathcal{O}^{\mu\nu} + \mathcal{O}^{\nu\mu} = 0$. Higher order corrections in the coupling constant α of the $U(1)$ subgroup are omitted. The induced electromagnetic current in (16) is a correction of order $1/m^4$ in the inverse heavy quark mass which vanishes in the limit of an infinitely heavy quark. Corrections in the inverse heavy quark masses are important for tests of the standard model at the present level of precision and have already been discussed in various areas of particle phenomenology [22–24].

3.2 Other types of induced currents

Expressions for the induced currents with quantum numbers other than that of the electromagnetic current $J^{PC} = 1^{--}$ can be obtained in a similar way. A review was recently presented in [25]. The axial current of fermions naturally appears in the standard model as the result of an axial-vector interaction of fermions with W and Z bosons. The corresponding induced current can be obtained as a derivative of the respective effective action with respect to the Z boson field. The leading order diagram, however, contains only two external legs and is ultraviolet (UV) divergent. In case of massless fermions, this diagram leads to the anomalous non-conservation of the axial current; that also requires a strict definition of the corresponding operator $\bar\psi\gamma_\mu\gamma_5\psi$ within perturbation theory, because its renormalization properties are not dictated by the Ward identities anymore in contrast to the vector current. The leading corrections to the anomaly of the axial current for massless fermions due to strong interactions were considered in [26,27] where the implications of the Adler–Bardeen no-go theorem (non-renormalizability) were discussed also for dimensional regularization and different definitions of the axial current at the tree level. Higher order corrections were analyzed in [28]. Explicit high-order corrections to the expression for the anomaly depend on the renormalization prescription for the composite operator $\bar\psi\gamma_\mu\gamma_5\psi$ within perturbation theory.

The scalar current $\bar\psi\psi$ appears in an interaction vertex for the Higgs boson and was intensively studied. Many terms of the heavy mass expansion for the scalar current were derived in [29]. The decay $H \to \gamma\gamma$ is described by the effective interaction

$$\Delta L_H = g_{\rm S}(m, \alpha_{\rm s})\alpha HFF, \tag{17}$$

with the effective local vertex $g_{\rm S}(m, \alpha_{\rm s})$ depending on the quark mass and the coupling constants (note the difference to $g_{\rm s}$, S stands for the scalar current). At the leading order the formfactor $g_{\rm S}(m, \alpha_{\rm s})$ is given by the corresponding one-loop diagram with two external photons. Here H is an interpolating field for the Higgs boson. The decay of the Higgs boson into two photons is calculated up to high orders of perturbation theory [30]. An application of the effective potential technique to the analysis of the correlators of the scalar gluonic currents which emerge in the decay of the Higgs boson into hadrons was considered in [31]. The mass expansion for the pseudoscalar current was also considered in some detail in relation to the intrinsic charm flavor of pseudoscalar light bosons [32].

3.3 Results for the induced tensor current

For completeness we discuss here the calculation of the induced (antisymmetric) tensor current interacting with photons in the context of the effective action. We consider a tensor current of the form

$$j^{\mu\nu} = \bar\psi\sigma^{\mu\nu}\psi, \quad \sigma^{\mu\nu} = \frac{\rm i}{2}[\gamma^\mu, \gamma^\nu] \tag{18}$$

and calculate its low-energy limit induced by a heavy quark loop. The properties of this current are rather similar to those of the electromagnetic current. Note that the classical vector mesons (ρ, ω, ϕ) interact with this current and can be produced by it. We introduce an interaction

$$\Delta L_T = g_T\bar\psi\sigma^{\mu\nu}\psi F_{\mu\nu} \tag{19}$$

in the Lagrangian of heavy quarks and readily find the effective action for gauge fields induced by such a vertex. The low-energy limit at the one-loop order reads

$$\Gamma_T = \frac{-g_T g_{\rm s}^3 d_{abc}}{6m^3(4\pi)^2}\left(2{\rm tr}(FG^aG^bG^c) - {\rm tr}(FG^a){\rm tr}(G^bG^c)\right), \tag{20}$$

with the same notations as in (6). According to the form of the effective interaction in (19) the induced current $J_{\mu\nu}$ is given by a derivative

$$g_T J^{\mu\nu} = -\frac{\delta\Gamma_T}{\delta F_{\mu\nu}}, \tag{21}$$

and explicitly reads

$$J^{\mu\nu} = \frac{-g_{\rm s}^3 d_{abc}}{6m^3(4\pi)^2}\left(2(G^aG^bG^c)^{\mu\nu} - (G^a)^{\mu\nu}{\rm tr}(G^bG^c)\right). \tag{22}$$

Note the lower power of the heavy quark mass in (22) as compared to (16). The induced current in (22) is of less phenomenological interest than the vector current, because there is no direct source for the tensor current in nature in contrast to the vector current which appears in e^+e^- annihilations. However, the calculation is simple and allows one to demonstrate some essential features of the technique. The technique of calculating in external fields is well developed. After integration over the heavy field ψ we obtain

$$J^{\mu\nu} \equiv \langle j^{\mu\nu}\rangle_\psi = {\rm tr}(\sigma^{\mu\nu}{\rm i}S(0, m; A)), \tag{23}$$

where the propagator of a heavy quark in the external field A is given by

$${\rm i}\langle T\psi(x)\bar\psi(0)\rangle_A = S(x, m; A), \tag{24}$$

and where by definition

$$S(0,m;A)=\int\frac{\mathrm{d}^Dp}{(2\pi)^D}S(p,m;A);\tag{25}$$

$S(p,m;A)$ is a propagator in momentum space. The propagator $S(p,m;A)$ can be expanded in powers of the external field A and its derivatives. The formal operator expression reads

$$S(p,m;A)=S(p,m)+S(p,m)\gamma^\mu A_\mu S(p,m)+\dots,\tag{26}$$

where in momentum space

$$S(p,m)\equiv S(p,m;0)=\frac{1}{m-\gamma^\mu p_\mu}\tag{27}$$

is a free field propagator. The convenient way for the calculation is to use the Fock–Schwinger (fixed-point) gauge for the external field,

$$x^\mu A_\mu(x)=0.\tag{28}$$

In this gauge the gauge field potential $A_\mu(x)$ can be expressed through the field strength tensor $G_{\mu\nu}$. For small values of x one has

$$A_\mu(x)=\frac{1}{2}x^\alpha G_{\alpha\mu}+\dots\tag{29}$$

Substituting this expression in (26), any factor x results in a derivative in p, and one obtains

$$A_\mu(p)=\frac{-\mathrm{i}}{2}G_{\alpha\mu}\partial^\alpha=\frac{\mathrm{i}}{2}G_{\mu\alpha}\partial^\alpha.\tag{30}$$

The piece of the expansion in (26) relevant to the tensor current calculation at the leading order reads

$$I(p,m;G)=S(p,m)\gamma^\alpha\frac{\mathrm{i}}{2}G_{\alpha\alpha'}\partial^{\alpha'}S(p,m)\gamma^\beta \times\frac{\mathrm{i}}{2}G_{\beta\beta'}\partial^{\beta'}S(p,m)\gamma^\gamma\frac{\mathrm{i}}{2}G_{\gamma\gamma'}\partial^{\gamma'}S(p,m).\tag{31}$$

All derivatives act to the right to all propagators that follow. Integrating this expression over p one obtains

$$I(m;G)=\int\frac{\mathrm{d}^Dp}{(2\pi)^D}I(p,m;G)\tag{32}$$

and

$$\langle j^{\mu\nu}\rangle_\psi=\mathrm{tr}(\sigma^{\mu\nu}\mathrm{i}I(m;G)).\tag{33}$$

The integral in (32) converges and the calculation can be performed in four-dimensional space-time, $D=4$. The calculation of consecutive derivatives on the right hand side of (31) is a bit cumbersome. A simplification can be obtained by applying the integration-by-parts technique to (32). This allows one to transfer the left derivative to the left propagator in (31) (the surface term vanishes), resulting in an equivalent expression for the integrand,

$$I(p,m;G)=-(\partial^{\alpha'}S(p,m))\gamma^\alpha\frac{\mathrm{i}}{2}G_{\alpha\alpha'}S(p,m)\gamma^\beta \times\frac{\mathrm{i}}{2}G_{\beta\beta'}\partial^{\beta'}S(p,m)\gamma^\gamma\frac{\mathrm{i}}{2}G_{\gamma\gamma'}\partial^{\gamma'}S(p,m).\tag{34}$$

The derivatives are readily calculable using

$$\partial^\alpha S(p,m)=S(p,m)\gamma^\alpha S(p,m),\tag{35}$$

which is just a Ward identity. This finally gives ($S=S(p,m)$)

$$I(p,m;G) =-\frac{\mathrm{i}}{2}G_{\alpha\alpha'}\frac{\mathrm{i}}{2}G_{\beta\beta'}\frac{\mathrm{i}}{2}G_{\gamma\gamma'}S\gamma^{\alpha'}S\gamma^\alpha S\gamma^\beta \times S(\gamma^{\beta'}S\gamma^\gamma S\gamma^{\gamma'}+\gamma^\gamma S\gamma^{\beta'}S\gamma^{\gamma'}+\gamma^\gamma S\gamma^{\gamma'}S\gamma^{\beta'})S.\tag{36}$$

Computing the trace and the integrals, we obtain (22). The only integral necessary here has the form (after Wick rotation into the Euclidean domain)

$$\int\frac{p^{2n}\mathrm{d}^4p}{(p^2+m^2)^7}=\pi^2B(n+2,5-n)(m^2)^{n-5},\tag{37}$$

where $B(x,y)$ is Euler's beta function. All angular integrations reduce to an averaging. With the result given by (22) we confirm the result of [25] which we were not aware of while writing the paper. For comparison we note the relation $4\mathrm{tr}(t^at^bt^c)=d^{abc}+\mathrm{i}f^{abc}$. Note that for the gauge group $SU(2)$ we have $d^{abc}=0$ which confirms the result of [25] found by a direct calculation using a symmetry transformation specific for $SU(2)$.

4 Phenomenological applications

High precision tests of the standard model remain one of the main topics of particle phenomenology [33]. The recent observation of a possible signal from the Higgs boson may complete the experimentally confirmed list of the standard model particles [34]. Because experimental data are becoming more and more accurate, the determination of numerical values of the parameters of the standard model Lagrangian will require more accurate theoretical formulae. Recently an essential development in high-order perturbation theory calculations has been observed. A remarkable progress has been made in the heavy quark physics where a number of new physical effects have been described theoretically with high precision. The cross section of top–antitop production near the threshold has been calculated at the next-to-next-to-leading order of an expansion in the strong coupling constant and velocity of a heavy quark with an exact account for the Coulomb interaction (as a review, see [35,36]). This theoretical breakthrough allows for the best determination of a numerical value of the top quark mass from the experimental data. The method of Coulomb resummation resides in a nonrelativistic approximation for the Green function of the quark–antiquark system near the threshold and has been successfully used for the heavy quark mass determination within sum rule techniques [37–39]. Being applied to quarkonium systems this method is considered to give the best estimates for the heavy quark mass parameters [40–43]. Technically an enhancement of near-threshold contributions to sum rules is achieved by considering integrals of the spectral density of the heavy quark production with weight functions which suppress the high-energy tail of the spectrum. The integrals with weight functions $1/s^n$ for different positive integer n, $s = E^2$, where E is the total energy of the quark–antiquark system, are called moments of the spectral density and are most often used in the sum rule analysis [44]. The interest in the precision determination of the c quark mass is especially high, because this parameter introduces the largest uncertainty to the theoretical calculation of the running electromagnetic coupling constant at M_Z which is one of the key quantities for the constraints to the Higgs boson mass [45]. We stress that in the corresponding determination of the running electromagnetic coupling constant at M_Z based on direct integration of the experimental data over the threshold region, the sensitivity of the results to the c quark mass is much weaker [46].

For phenomenological applications of our results one therefore has to compute the two-point correlation functions of the induced currents. In the following we show that there is a strong constraint on the order n of the moment that can be used in heavy quark sum rules. Because of the contribution of low-energy gluons, only the first few moments formally exist if the theoretical expressions for the correlators include the $O(\alpha_s^3)$ order of perturbation theory.

4.1 The spectrum for the induced vector current correlator

First we discuss the case of the vector current where the data are obtained from e^+e^- annihilation experiments and are rather precise. The basic quantity for the analysis of a vector current $j^\mu = \bar\psi\gamma^\mu\psi$ of a heavy fermion ψ within sum rules is the vacuum polarization function

$$12\pi^2\mathrm{i}\int\langle Tj_\mu(x)j_\nu(0)\rangle e^{\mathrm{i}qx}\mathrm{d}^4x = (q_\mu q_\nu - g_{\mu\nu}q^2)\Pi(q^2). \tag{38}$$

With the spectral density $\rho(s)$ defined by the relation

$$\rho(s) = \frac{1}{2\pi\mathrm{i}}(\Pi(s+\mathrm{i}0) - \Pi(s-\mathrm{i}0)), \quad s>0, \tag{39}$$

the dispersion representation

$$\Pi(q^2) = \int\frac{\rho(s)\mathrm{d}s}{s-q^2} \tag{40}$$

holds. A necessary regularization and subtraction is assumed in (40). The normalization of the vacuum polarization function $\Pi(q^2)$ in (38) is chosen so that one obtains the high-energy limit $\lim_{s\to\infty}\rho(s) = 1$ for a lepton. For the quark in the fundamental representation of the gauge group $SU(N_c)$ the high-energy limit of the spectral density reads $\rho(\infty) = N_c$. The integral in (40) runs over the whole spectrum of the correlator in (38) or over the whole support of the spectral density $\rho(s)$ in (39).

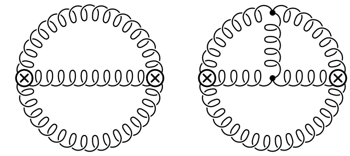

Fig. 2a,b. Induced massless correlator diagrams

A correlator of the induced vector current J^μ has the general form

$$\langle TJ^\mu(x)J^\nu(0)\rangle = -\partial_\alpha\partial_\beta\langle T\mathcal{O}^{\mu\alpha}(x)\mathcal{O}^{\nu\beta}(0)\rangle, \tag{41}$$

where an explicit expression of the current as a derivative of the antisymmetric operator $\mathcal{O}^{\mu\nu}$ has been employed. The resulting correlator $\langle T\mathcal{O}^{\mu\alpha}(x)\mathcal{O}^{\nu\beta}(0)\rangle$ in (41) contains only gluonic operators. Such correlators were considered previously in the framework of perturbation theory [47,48]. In leading order of perturbation theory the correlator in (41) has the topological structure of a sunset diagram, as is shown in Fig. 2a. Technically, a convenient procedure of computing the sunset-type diagrams is to work in configuration space [49]. We find

$$\langle TJ_\mu(x)J_\nu(0)\rangle = \frac{-34d_{abc}d_{abc}}{2025\pi^4m^8}\left(\frac{\alpha_s}{\pi}\right)^3\left(\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2\right)\frac{1}{x^{12}}. \tag{42}$$

A Fourier transform of the correlator in (42) gives the vacuum polarization function in momentum space, which reads

$$12\pi^2\mathrm{i}\int\langle TJ_\mu(x)J_\nu(0)\rangle e^{\mathrm{i}qx}\mathrm{d}^4x = (q_\mu q_\nu - g_{\mu\nu}q^2)\Pi(q^2), \tag{43}$$

where at small q^2 $(q^2 \ll m^2)$

$$\Pi(q^2)|_{q^2\approx 0} = C_g\left(\frac{q^2}{4m^2}\right)^4\ln\left(\frac{\mu^2}{-q^2}\right), \quad C_g = \frac{17d_{abc}d_{abc}}{243000}\left(\frac{\alpha_s}{\pi}\right)^3. \tag{44}$$

For QCD with color group $SU(3)$ one has $d_{abc}d_{abc} = 40/3$. The spectral density of the vacuum polarization function $\Pi(q^2)$ in (43) is given at small values for s by

$$\rho(s)|_{s\approx 0} = C_g\left(\frac{s}{4m^2}\right)^4. \tag{45}$$

Note that the spectral density given in (45) can be found without an explicit calculation of the Fourier transform of the correlator in (43). Instead of computing the Fourier transform one can use a spectral decomposition (dispersion representation) in configuration space which was heavily employed for the analysis of sunset diagrams in [49]. In this particular instance the spectral representation of the correlator in configuration space reads

$$\frac{\mathrm{i}}{x^{12}} = \frac{\pi^2}{2^8\Gamma(6)\Gamma(5)}\int_0^\infty s^4D(x^2,s)\mathrm{d}s, \tag{46}$$

with $D(x^2, s)$ being the propagator of a scalar particle of mass $s^{1/2}$,

$$D(x^2, m^2) = \frac{im\sqrt{-x^2}K_1(m\sqrt{-x^2})}{4\pi^2(-x^2)}, \quad (47)$$

where $K_1(z)$ is the McDonald function (a modified Bessel function of the third kind; see e.g. [50]). $\Gamma(z)$ is Euler's gamma function.

An asymptotic behavior of the spectral density of the corresponding contribution for large energies (where the limit of massless quarks can be used) enters the expression for the ratio $R(s)$ of the e^+e^- annihilation into hadrons and has been known since long ago [51–53]. This term is usually called a light-by-light (lbl) contribution and reads

$$R^{\rm lbl}(s) = \left(\frac{\alpha_{\rm s}}{\pi}\right)^3 \frac{d_{abc}d_{abc}}{1024}\left(\frac{176}{3} - 128\zeta(3)\right). \quad (48)$$

Here $\zeta(z)$ is the Riemann ζ function with $\zeta(3) = 1.20206\ldots$ The contribution to the spectral density given in (48) is negative, while our result given in (45) is positive as it should be for the spectral density of the electromagnetic current as a Hermitean operator.

4.2 The spectrum for the induced tensor current correlator

The results for the correlator of the tensor current given in (22) are slightly more complicated. The correlator reads

$$\begin{aligned} &12\pi^2{\rm i}\int\langle TJ_{\mu\nu}(x)J_{\alpha\beta}(0)\rangle {\rm e}^{{\rm i}qx}{\rm d}^4x \\ &= (g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha})\Pi_g(q^2) \quad (49)\\ &+(g_{\mu\alpha}q_\nu q_\beta - g_{\mu\beta}q_\nu q_\alpha - g_{\nu\alpha}q_\mu q_\beta + g_{\nu\beta}q_\mu q_\alpha)\Pi_q(q^2), \end{aligned}$$

where two scalar amplitudes are possible now. With the explicit expressions for the induced tensor current $J_{\mu\nu}$ in (22) one finds

$$\begin{aligned} \Pi_g(q^2) &= -\frac{q^2}{2}\Pi_q(q^2) \quad (50)\\ &= \frac{d_{abc}d_{abc}}{3240}\left(\frac{\alpha_{\rm s}}{\pi}\right)^3 \frac{-q^2}{4}\left(\frac{q^2}{4m^2}\right)^3 \ln\left(\frac{\mu^2}{-q^2}\right). \end{aligned}$$

The physical content of the amplitudes $\Pi_g(q^2)$ and $\Pi_q(q^2)$ is related to contributions of the states with $J^{PC} = 1^{--}$ and $J^{PC} = 1^{+-}$, respectively. Note that the sum rule analysis for the mesons with quantum numbers $J^{PC} = 1^{+-}$ has been done in [54] with quark interpolating currents. Based on the present results we also see a possibility to use gluonic currents as the interpolating operators for such mesons. The validity of such a description depends strongly on the strength of the interaction of the meson in question with the corresponding interpolating operator which is difficult to estimate independently.

Note that there are only two independent gluonic operators available to construct the induced currents under consideration. The electromagnetic current is given by a derivative of a special linear combination of these operators while the tensor current is given by a linear combination of the operators themselves. There is one more current relevant to the situation. It originates from the Gordon decomposition of the electromagnetic current (see e.g. [19])

$$2m\bar\psi\gamma^\mu\psi = \partial_\nu(\bar\psi\sigma^{\mu\nu}\psi) + \bar\psi{\rm i}\overleftrightarrow{D}^\mu\psi, \quad \overleftrightarrow{D} = \overrightarrow{D} - \overleftarrow{D}. \quad (51)$$

This relation holds for the induced currents as well. The left hand side and the right hand side of (51) have different parity as for the number of Dirac γ-matrices between spinor fields which is reflected in an additional factor m at the left hand side of (51). In the massless limit these types of currents are alien and can never mix. At the level of induced currents the Dirac structure of the initial heavy quark currents is reflected in different degrees of suppression by the heavy quark mass m.

4.3 The spectrum for a mixed current correlator

Having both vector and tensor induced currents at hand, one can study a mixed correlator of the form

$$\begin{aligned} &12\pi^2{\rm i}\int\langle TJ_\mu(x)J_{\alpha\beta}(0)\rangle {\rm e}^{{\rm i}qx}{\rm d}^4x \\ &= {\rm i}(g_{\mu\alpha}q_\beta - g_{\mu\beta}q_\alpha)\Pi_M(q^2), \quad (52)\end{aligned}$$

with a single scalar amplitude $\Pi_M(q^2)$. Such mixed correlators are useful in sum rule applications [55]. One finds

$$\Pi_M(q^2) = \frac{d_{abc}d_{abc}}{8100}\left(\frac{\alpha_{\rm s}}{\pi}\right)^3 \frac{-q^2}{4m}\left(\frac{q^2}{4m^2}\right)^3 \ln\left(\frac{\mu^2}{-q^2}\right). \quad (53)$$

The physical content of the amplitude $\Pi_M(q^2)$ is given by the $J^{PC} = 1^{--}$ resonances, i.e. by the Υ meson family in case of b quarks for the original currents and by the ρ meson family in case of induced currents at low energies.

5 Moments of the spectral density

As mentioned above, the moments of the spectral density $\rho(s)$ of the form

$$\mathcal{M}_n = \int\frac{\rho(s){\rm d}s}{s^{n+1}} \quad (54)$$

are usually studied within the sum rule method for heavy quarks [44]. These moments are related to the derivatives of the vacuum polarization function $\Pi(q^2)$ at the origin,

$$\mathcal{M}_n = \frac{1}{n!}\left(\frac{\rm d}{{\rm d}q^2}\right)^n \Pi(q^2)\bigg|_{q^2=0}. \quad (55)$$

Such moments are chosen in order to suppress the high-energy part of the spectral density $\rho(s)$ which is not measured accurately in the experiment. Within the sum rule method one assumes that the moments in (54) can be calculated for any n or, equivalently, that the derivatives in (55) exist for any n. The existence of these moments seems to be obvious, because one implicitly assumes that the spectral density $\rho(s)$ of the current correlator of the heavy quarks with mass m vanishes below the two-particle threshold at $s = 4m^2$. This implicit assumption means that the vacuum polarization function $\Pi(q^2)$ of heavy quarks is analytic in the whole complex plane of q^2, except for the cut along the positive real axis starting from $4m^2$. This assumption about the analytic properties of the vacuum polarization function $\Pi(q^2)$ is known to be wrong if a resummation of Coulomb effects to all orders of perturbation theory is performed: as a result of such a resummation the Coulomb bound states appear below the perturbation theory threshold at $s = 4m^2$.

5.1 Infrared singular behavior of the moments

The qualitatively new feature of the effective currents given in (16) and (22) is that they are expressed through massless fields. Therefore, the spectrum of the two-point correlators of these currents starts at zero energy (at least for finite orders of perturbation theory). This feature drastically changes the analytic structure of the two-point correlators of these currents and, in particular, their infrared (IR) or small q^2 behavior because of the branching point (cut) singularity of $\Pi(q^2)$ at the origin $q^2 = 0$. This new feature of having a nonvanishing spectrum below the formal tree-level two-particle threshold appearing at the $O(\alpha_s^3)$ order of perturbation theory for induced current correlators has important phenomenological consequences. Indeed, such a change of the analytic structure of induced current correlators strongly affects the theoretical expressions for some observables usually employed in heavy quark physics for the precision determination of the parameters of heavy quarks and their interactions.

Because of the low-energy gluon contributions, the large n moments of the spectral density in (54) related to derivatives of the correlator at the origin do not exist and cannot be directly used for phenomenological analyses. Caused by the factor $(s/4m^2)^4$ in (54), the moments become IR singular for $n \geq 4$ in case of the induced vector current. This can already be seen by looking at the factor $1/m^4$ in the induced vector current in (16). For the induced tensor current the corresponding moments start to diverge earlier because of a weaker suppression by the heavy quark mass; the corresponding factor in (22) is $1/m^3$ instead of $1/m^4$. Therefore, in this case the moments become IR singular already for $n \geq 3$. Note that in original applications of sum rules quite large ns were used. For instance, the sum rule analysis of charmonium ($c\bar{c}$ system) is usually performed at $n \sim 3 \div 7$. These moments were used for extracting the numerical value of the gluon condensate from sum rules [44,56]. In the precision analysis of the $b\bar{b}$ system the contribution of the gluon condensate is relatively smaller because the b quark is heavier than the c quark. Thus, the gluon condensate contribution is stronger suppressed by the heavy quark mass as compared to the perturbation theory contribution and higher order moments were used in the analysis of the $b\bar{b}$ system. In different papers on this subject moments in the range $n \sim 3 \div 20$ were considered. In view of our result on the low-energy behavior of the spectral density, one has either to limit the accuracy of theoretical calculations for the standard moments to the $O(\alpha_s^2)$ order of perturbation theory, which seems insufficient for a high precision analysis of quarkonium systems (especially for $b\bar{b}$ with the Coulomb resummation performed to all orders) or to use only a few first moments. For small values of n, however, the high-energy contribution, which is not known experimentally with a reasonable precision, is not sufficiently suppressed and introduces a large quantitative uncertainty into the sum rules for the moments. An analysis based on finite energy sum rules is free from such a problem and can be used in phenomenological applications [57].

Note in passing that there is no low-energy gluon contribution (and therefore no low-energy divergence problem or a non-analyticity at the origin) for correlators of the currents containing only one heavy quark with mass m. The spectrum of such correlators starts at m^2 and there are no massless intermediate states contributing to the correlator in the perturbation theory of strong interactions (see e.g. [58]). The theoretical expressions for such correlators can be used for high precision tests of theoretical predictions when the accuracy of experimental data in corresponding channels will improve in the future.

5.2 An infrared safe definition for the moments

The infinite-integration sum rules with large n can be retained at high orders of perturbation theory if an appropriate cutoff at small energies is introduced. This can be readily achieved by calculating the moments (or derivatives) at some Euclidean point $q^2 = -\Delta$ [59]. Indeed, for the regularized moments

$$\mathcal{M}_n(\Delta) = \frac{1}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}q^2}\right)^n \Pi(q^2)\bigg|_{q^2=-\Delta} = \int_0^\infty \frac{\rho(s)\mathrm{d}s}{(s+\Delta)^{n+1}} \tag{56}$$

there is no divergence at small s. However, the regularization parameter Δ cannot be arbitrary small. The reason is that the resulting correlator of gluonic currents in (41) is essentially normalized at $\mu^2 = \Delta$ when radiative corrections are taken into account. The relevant diagrams are shown in Fig. 2b. The radiative corrections are known to be large and, therefore, Δ should be larger than the expected IR scale in the quark channels [47]. This observation makes the phenomenological analysis based on the regularized sum rules in (56) unprecise even for reasonably large values of n because the continuum contribution to moments is not sufficiently suppressed for large values of Δ. The suppression of the high-energy tail of the spectral density can be enhanced by constructing a more special kind of moments which exploit the explicit behavior of the spectral density at low energy. The IR safe moments with a more efficient suppression of the high-energy tail for the induced vector current can be chosen in the form

$$\tilde{\mathcal{M}}_n(\Delta) = \int_0^\infty \frac{\rho(s)\mathrm{d}s}{s^3(s+\Delta)^{n+1}}. \quad (57)$$

While the formal statement about the analytic properties of heavy quark correlators at the origin determined by the massless intermediate states is rather straightforward, the main issue is whether the low-energy gluon contribution is essential for phenomenology. In this respect we would like to remind the reader that Coulomb poles which are essential for the analysis of the Υ resonance and the $b\bar{b}$ production near threshold give contributions to moments which are formally of the $O(\alpha_s^3)$ order (for the value of the Coulombic wave function at the origin see e.g. [40]). This order coincides with the order of corrections considered here (see (44)). The theoretical expressions for the correlators in the scalar channel where the below-threshold corrections start at the $O(\alpha_s^2)$ order are more sensitive to these special below-threshold contributions. However, data in the scalar channel are considerably worse than those in the vector channel, and there is no possibility of a high precision analysis in the scalar channel at present.

We give some quantitative estimates of relative contributions of low-energy gluon states to the IR safe moments given in (56). For the contribution of the low-energy gluon term to the regularized moments we find

$$\mathcal{M}_n^{\rm leg}(\Delta) = \frac{1}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}q^2}\right)^n \Pi_{\rm leg}(q^2)\Big|_{q^2=-\Delta} = \frac{C_g}{(4m^2)^4}\frac{24\Gamma(n-4)}{n!\Delta^{n-4}}. \quad (58)$$

The derivatives can be explicitly calculated for $n \geq 5$ with the relation

$$\left(\frac{\mathrm{d}}{\mathrm{d}q^2}\right)^n (q^2)^4 \ln\left(\frac{\mu^2}{-q^2}\right) = \frac{24\Gamma(n-4)}{(-q^2)^{n-4}}. \quad (59)$$

For smaller n there is a dependence on the UV cutoff μ^2; we consider this case later. Note that the contribution of low-energy gluons into the derivatives at the point $q^2 = -\Delta$ is calculated by direct differentiation of the expression $\Pi_{\rm leg}(q^2)$ for the low-energy limit of the correlator $\Pi(q^2)$. The same result (in a pure mathematical sense) is obtained with the dispersion relation

$$\frac{1}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}q^2}\right)^n \Pi_{\rm leg}(q^2)\Big|_{q^2=-\Delta} = \int_0^\infty \frac{\rho_{\rm leg}(s)\mathrm{d}s}{(s+\Delta)^{n+1}}, \quad (60)$$

where $\rho_{\rm leg}(s)$ is the spectral density given in (45).

The leading order moments for the contribution from the two-particle threshold read

$$\mathcal{M}_n^0(\Delta) = \frac{1}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}q^2}\right)^n \Pi_0(q^2)\Big|_{q^2=-\Delta} = \int_{4m^2}^\infty \frac{\rho_0(s)\mathrm{d}s}{(s+\Delta)^{n+1}}$$

$$= \frac{4m^2 N_c}{(4m^2+\Delta)^{n+1}}\frac{\Gamma(5/2)\Gamma(n)}{2\Gamma(n+5/2)}\Bigg[(2n+3)\,{}_2F_1\left(3/2,n+1;n+3/2;\frac{\Delta}{4m^2+\Delta}\right) - {}_2F_1\left(5/2,n+1;n+5/2;\frac{\Delta}{4m^2+\Delta}\right)\Bigg], \quad (61)$$

with

$$\rho_0(s) = N_c\sqrt{1-\frac{4m^2}{s}}\left(1+\frac{2m^2}{s}\right). \quad (62)$$

Here ${}_2F_1(\ldots)$ is a hypergeometric function. For order-of-magnitude estimates we put $\Delta = 0$ in these regular moments, which is a good approximation and allows for having a simple analytical formula, given by

$$\mathcal{M}_n^0(0) = \frac{N_c}{(4m^2)^n}\frac{\Gamma(n)\Gamma(5/2)}{\Gamma(n+5/2)}(n+1). \quad (63)$$

For $\alpha_s = 0.3$, $m = m_b = 4.8\,\mathrm{GeV}$ and $\Delta = 1\,\mathrm{GeV}^2$ we find that the low-energy gluon term in (58) is larger than the standard leading order term in (61) for $n > 7$. Note that α_s should be normalized at a low scale, and our choice corresponds to the normalization at the scale of the τ lepton mass, $M_\tau = 1.777\,\mathrm{GeV}$. Therefore, for the $b\bar{b}$ system, i.e. for the analysis of Υ resonances the dependence on the cutoff Δ is numerically essential for moments at large $n > 7$ and for Δ as large as $\Delta = 1 \div 2\,\mathrm{GeV}^2$, which is still small according to the estimates of radiative corrections in gluonic channels.

In the case of the c quark the contribution of low-energy gluons has no chance to become large because other corrections (like the gluon condensate) become more important for large n. Therefore, the low-energy gluon term is numerically negligible in the range of values $n < 7 \div 8$ which can be taken in order that the nonperturbative expansion within the operator product expansion is still valid for the charm quark current correlator. In this range for n the correction due to the low-energy gluon term is not essential for the $c\bar{c}$ system, i.e. for analyzing the J/ψ resonances.

The above crude estimate shows that the contribution of the low-energy gluons is large for values of parameters usually taken in the b quark physics analysis and should therefore be taken into account. In Table 1 we give a more detailed quantitative analysis in numerical form. We see

Table 1. The ratios of the moments from different squared energy regions for the bottom quark ($m_b = 4.8\,\mathrm{GeV}$) with $E_R = 1\,\mathrm{GeV}$ and two different values for Δ

	$\Delta = 1\,\mathrm{GeV}^2$		$\Delta = 2\,\mathrm{GeV}^2$	
	$\mathcal{M}_n^{(a)}/\mathcal{M}_n^{(c)}$	$\mathcal{M}_n^{(b)}/\mathcal{M}_n^{(c)}$	$\mathcal{M}_n^{(a)}/\mathcal{M}_n^{(c)}$	$\mathcal{M}_n^{(b)}/\mathcal{M}_n^{(c)}$
$n=5$	0.000	0.854	0.000	0.841
$n=6$	0.002	1.151	0.001	1.133
$n=7$	0.085	1.507	0.011	1.481
$n=8$	4.173	1.932	0.280	1.897
$n=9$	239.3	2.439	8.089	2.392

that the moments are given by the integral over the spectrum for the whole range $(0,\infty)$. We represent the integral as a sum of three pieces. The first one gives the integral from $s=0$ to $s=4m^2$, where we use $\rho_{\rm leg}(s)$ as a spectral density (note the bold extrapolation to energies as large as $2m$), the second is an integral over the resonance region from $s=4m^2$ to $s=(2m+E_R)^2$ where E_R is of order 1 GeV for the $b\bar b$ system, and the last one is the continuum contribution from $s=(2m+E_R)^2$ to $s=\infty$. So we consider

$$\mathcal{M}_n^{(a)}=\int_0^{4m^2}\frac{\rho_{\rm leg}(s){\rm d}s}{(s+\Delta)^{n+1}},$$
$$\mathcal{M}_n^{(b)}=\int_{4m^2}^{(2m+E_R)^2}\frac{\rho_0(s){\rm d}s}{(s+\Delta)^{n+1}},$$
$$\mathcal{M}_n^{(c)}=\int_{(2m+E_R)^2}^{\infty}\frac{\rho_0(s){\rm d}s}{(s+\Delta)^{n+1}}. \qquad (64)$$

It is convenient to normalize the first two contributions $\mathcal{M}_n^{(a)}$ and $\mathcal{M}_n^{(b)}$ to the continuum contribution $\mathcal{M}_n^{(c)}$. The result for the choice $\Delta=1\,{\rm GeV}^2$ is shown in the second and third column in Table 1. It is apparent that for $n=5,6$ the contribution of the resonance region is about equal to the contribution of the continuum, a fact which is necessary for a precision calculation of experimental moments. For larger values of n, in the range of $n>7$, the resonance region dominates. However, in this region the contribution of the low-energy gluon term is large and comparable with the resonance contribution. For even larger n the contribution of the low-energy gluon term is very big. The situation is softer for $\Delta=2\,{\rm GeV}^2$, as shown in the fourth and fifth column in Table 1. Note that the contribution of the gluon condensate for the $b\bar b$ system is negligible up to high values, $n<20$. Thus, the high-order derivatives at $q^2=-\Delta$ are saturated by the closest singularity (the cut from the origin) and this singularity becomes dominant for large values of n despite the strong numerical suppression by α_s^3 and the small numerical factor given by C_g in (45).

Note that the inclusion of such a diagram with a below-threshold cut requires an accurate interpretation of the data. Indeed, if moments are formally calculated as derivatives, their relation to the spectrum is not explicit. As we have seen, the diagram leading to the low-energy gluon term gives a large contribution to the correlator, which cannot be neglected. As far as a comparison with specific data is concerned it is clear that the near-origin contributions are accounted for by the effective low-energy theory and should be subtracted properly from the b quark production. Thereby, the correct correspondence of relevant data to the theoretical spectrum can be restored, i.e. the theoretical moments for a specific set of data can be properly adjusted by specifying the diagrams included. Our analysis shows that this careful procedure of comparison is not an academic question but a practical necessity at the present level of precision in the $b\bar b$ production. For the modified moments $\widetilde{\mathcal{M}}_n(\Delta)$ in (57) the picture is qualitatively the same with some difference as for the range of Δ and n where the soft gluon corrections are important.

As a concluding remark, let us note that one can definitely introduce observables as integrals starting from $4m^2$ or some other given point $s_{\rm thr}$,

$$\int_{s_{\rm thr}}^{\infty}\frac{\rho(s){\rm d}s}{s^{n+1}}. \qquad (65)$$

These moments are suitable observables for sufficiently large values of $s_{\rm thr}$, but they are not related to derivatives of the polarization function at the origin. Instead, they are related to a IR modified correlator of the form

$$\Pi^{\rm IR}(q^2)=\Pi(q^2)-\int_0^{s_{\rm thr}}\frac{\rho(s){\rm d}s}{s-q^2}, \qquad (66)$$

which is analytic at the origin. For small s the spectral density $\rho(s)$ in (66) is well approximated by the low-energy gluon spectral density $\rho_{\rm leg}(s)$. This observation on the analytic structure of the correlator at low energies is important for a nonperturbative calculation of the derivatives of the polarization function at the origin (for instance, in a future computation on the lattice with dynamic quarks) which can be used for a comparison with experimental data on the $b\bar b$ production. Note that in a sense the IR subtraction in (66) is equivalent to a redefinition of the heavy quark current: the operator $j^\mu_{b\bar b}$ relevant to the valence bottom quark pair production should not contain the spectrum below some IR cutoff, i.e.

$$j^\mu_{b\bar b}=\bar b\gamma^\mu b-J^\mu. \qquad (67)$$

The precise meaning of this expression is explained in (66).

6 Conclusions

We have presented explicit low-energy expressions for the induced currents of heavy quarks. The induced currents are expressed through the gluon operators generated by a virtual heavy quark loop. We have considered the electromagnetic current and the tensor current closely related to it. Heavy quark loop induced corrections to the correlators at low energies (below the formal two-particle threshold) first appear at $O(\alpha_s^3)$ order of perturbation theory and are given by the $1/m^4$ term of the mass expansion in case of the electromagnetic current and by the $1/m^3$ term in case of the tensor current. The spectra of the correlators of such induced currents start at zero energy. This fact makes impossible the standard analysis of the moment sum rules related to derivatives of the polarization function at the origin at $O(\alpha_s^3)$ order of perturbation theory for $n\ge 4$ in case of the electromagnetic current and for $n\ge 3$ in case of the tensor current. The contribution of the low-energy gluons is numerically significant for b quarks and should properly be taken into account for the high precision analysis of $b\bar b$ production.

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